Reciprocals of Integers

As a decimal, 1/2 = 0.5 and 1/3 = 0.33333... One comes to a stop, and the other goes forever repeating 3.

When you express 1/7 as a decimal, does it come to a stop or go on forever? Does it repeat a pattern, or never repeat (as with π = 3.14159265358979462643 38....)? Let's divide 7 into 1 by long division to get clues.

```
  0.142857142857…
7)1.000000000000…
  7
  30
  28
   20
   14
    60
    56
     40
     35
      50
      49
       1
```

With a remainder of 1 again, the pattern 142857 must repeat. How long can the repeating pattern be? There are only six possible remainders: **1**, **2**, **3**, **4**, **5**, **6**. So we can't go more than 6 remainders without repeating one, and maximum length of the repeating pattern is 6. In this case the reciprocal actually had the maximum length. Is that unusual? How many other reciprocals do that? For instance, does the decimal for 1/11 go10 before repeating?

1/7 = 0.142857142857142857142857... is a repeating decimal with a period of 7 – 1 = 6.
For what other n is $1/n$ a repeating decimal with period $n - 1$?

1/2 = 0.5
1/3 = 0.3333333333333333333333333333333333...
1/4 = 0.25
1/5 = 0.2
1/6 = 0.166666666666666666666666666666666666...
1/7 = 0.142857142857142857142857142857142857...
1/8 = 0.125
1/9 = 0.111111111111111111111111111111111111...
1/10 = 0.1
1/11 = 0.090909090909090909090909090909090909...
1/12 = 0.083333333333333333333333333333333333...
1/13 = 0.076923076923076923076923076923076923...
1/14 = 0.071428571428571428571428571428571428...
1/15 = 0.066666666666666666666666666666666666...
1/16 = 0.0625
1/17 = 0.0588235294117647058823529411764705 88...
1/18 = 0.055555555555555555555555555555555555...
1/19 = 0.052631578947368421052631578947368421...
1/20 = 0.05
1/21 = 0.047619047619047619047619047619047619...
:
1/97 = ?

Expressing "repeating decimal" as an equation

$$1/7 = 0.142857142857142857142857...$$

Repeating decimal with a period of 6. This can be expressed as

$$1/7 = 142657 / 999999 \qquad \text{Eq.1}$$

Note that $1/999999 = 0.000001000001000001...$

$$142857 \times 0.000001000001000001... = 0.142857142857142857142857...$$

Rearranging Eq.1 we get $999999/7 = 142857$
or $(10^{7-1} - 1)/7 = 142857$ (an integer)

So our question, "How many reciprocals $1/n$ repeat with a pattern of length $n - 1$?"
This is the same as asking "For what ns is $(10^{n-1}-1)/n$ = integer?

$1/n$	$(10^{n-1} - 1)/n$
$1/2 = 0.5$	$(10^1-1)/2 = 4.5$
$1/3 = 0.333333333333333333333333333333333333...$	$(10^2-1)/3 = 33$
$1/4 = 0.25$	$(10^3-1)/4 = 249.75$
$1/5 = 0.2$	$(10^4-1)/5 = 1999.8$
$1/6 = 0.166666666666666666666666666666666666...$	$(10^5-1)/6 = 16666.5.$
$1/7 = 0.142857142857142857142857142857142857...$	$(10^6-1)/7 = 142857$
$1/8 = 0.125$	$(10^7-1)/8 = 1249999.875$
$1/9 = 0.111111111111111111111111111111111111...$	$(10^8-1)/9 = 11111111$
$1/10 = 0.1$	$(10^9-1)/10 = 99999999.9$
$1/11 = 0.090909090909090909090909090909090909...$	$(10^{10}-1)/11 = 909090909$
$1/12 = 0.083333333333333333333333333333333333...$	$(10^{11}-1)/12 = 8333333333.25$
$1/13 = 0.076923076923076923076923076923076923...$	$(10^{12}-1)/13 = 76923076923$
$1/14 = 0.071428571428571428571428571428571428...$	$(10^{13}-1)/14 = 714285714285.642857...$
$1/15 = 0.066666666666666666666666666666666666...$	$(10^{14}-1)/15 = 6666666666666.6$
$1/16 = 0.0625$	$(10^{15}-1)/16 = 62499999999999.9375$
$1/17 = 0.058823529411764705882352941176470588...$	$(10^{16}-1)/17 = 588235294117647$
$1/18 = 0.055555555555555555555555555555555555...$	$(10^{17}-1)/18 = 5555555555555555.5$
$1/19 = 0.052631578947368421052631578947368421...$	$(10^{18}-1)/19 = 52631578947368421$
$1/20 = 0.05$	$(10^{19}-1)/20 = 499999999999999999.95$
$1/21 = 0.047619047619047619047619047619047619...$	$(10^{20}-1)/21 = 4761904761904761904.714285...$

Does this happen in only the decimal (base-10) number system? Let's look at base-2.

$1/n$ (right-hand side in base-2)	$(2^{n-1}-1)/n$
1/2 = 0.100000000000000000000000000000000...	$(2^1-1)/2 = 1/2 = 0.5$
1/3 = 0.010101010101010101010101010101010101...	$(2^2-1)/3 = 3/3 = 1$
1/4 = 0.010000000000000000000000000000000000...	$(2^3-1)/4 = 7/4 = 1.75$
1/5 = 0.001100110011001100110011001100110011...	$(2^4-1)/5 = 15/5 = 3$
1/6 = 0.001010101010101010101010101010101010...	$(2^5-1)/6 = 31/6 = 5.1666...$
1/7 = 0.001001001001001001001001001001001001...	$(2^6-1)/7 = 63/7 = 9$
1/8 = 0.001000000000000000000000000000000000...	$(2^7-1)/8 = 127/8 = 15.875$
1/9 = 0.000111000111000111000111000111000111...	$(2^8-1)/9 = 255/9 = 28.3333...$
1/10 = 0.000110011001100110011001100110011001...	$(2^9-1)/10 = 511/10 = 51.1$
1/11 = 0.000101110100010111010001011101000101...	$(2^{10}-1)/11 = 1023/11 = 93$
1/12 = 0.000101010101010101010101010101010101...	$(2^{11}-1)/12 = 2047/12 = 170.5833...$
1/13 = 0.000100111011000100111011000100111011...	$(2^{12}-1)/13 = 4095/13 = 315$
1/14 = 0.000100100100100100100100100100100100...	$(2^{13}-1)/14 = 8191/14 = 585.07142...$
1/15 = 0.000100010001000100010001000100010001...	$(2^{14}-1)/15 = 16383/15 = 1092.2$
1/16 = 0.000100000000000000000000000000000000...	$(2^{15}-1)/16 = 32767/16 = 2047.9375$
1/17 = 0.000011110000111100001111000011110000...	$(2^{16}-1)/17 = 65535/17 = 3855$
1/18 = 0.000011100011100011100011100011100011...	$(2^{17}-1)/18 = 131071/18 = 7281.72222...$
1/19 = 0.000011010111100101000011010111100101...	$(2^{18}-1)/19 = 262143/19 = 13797$
1/20 = 0.000011001100110011001100110011001100...	$(2^{19}-1)/20 = 524287/20 = 26214.35$
1/21 = 0.000011000011000011000011000011000011...	$(2^{20}-1)/21 = 1048575/21 = 49932.142857...$

Observations:

$(10^{n-1}-1)/n$ = integer for n prime, except for $n = 2$ and $n = 5$.

$(10^{n-1}-1)/n$ = integer for $n = 9$ (not prime).

$(2^{n-1}-1)/n$ = integer for n prime, except for $n = 2$.

Conjecture:

$(a^{n-1}-1)/n$ = integer for n prime, except when n is a factor of a.

The Modulo Operation

We need to turn the informal equation $(a^{n-1}-1)/n$ = integer into a formal equation.

m modulo n (abbreviated m mod n) is the remainder after the integer n is divided into the integer m a whole number of times. For example,

30 mod 7 = 2 because 7 goes into 30 four times with 2 left over.
51 mod 17 = 0 because 17 goes into 51 three times with nothing left over.

So saying "m/n = integer" is the same as saying "m mod n = 0"

Restatement of Conjecture:

$(a^{n-1}-1)$ mod $n = 0$ for n prime, except when n is a factor of a,
or
a^{n-1} mod $n = 1$ for n prime, except when n is a factor of a.

This is "Fermat's Little Theorem."

Pseudoprimes

Fermat's Little Theorem says

if n is prime, then a^{n-1} mod $n = 1$.

But it is not always true that

if a^{n-1} mod $n = 1$, then n is prime.

For example, 10^{9-1} mod 9 = 1, even though $9 = 3 \cdot 3$ (not prime).
Another example: 2^{341-1} mod 341 = 1, even though $341 = 11 \cdot 31$ (not prime).
341 is called a "Fermat pseudoprime" base-2.
But 341 is not a pseudoprime base-3 because 3^{341-1} mod 341 = 56 (not 1).

Some numbers are pseudoprimes for many bases. For example, 91 is a pseudoprime base-36, -40, -61 , -66, -75, -79, -82, -87, -88, and -90.