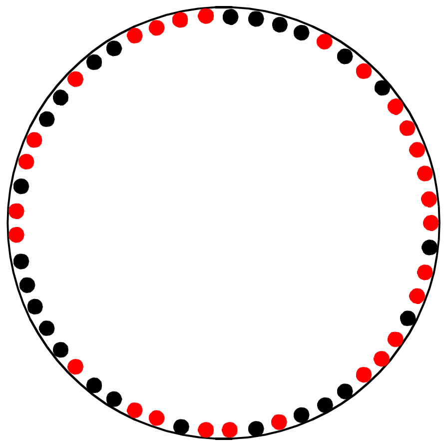
Unique binary patterns for plates

$n = 53$ (number of bits)

$2^n = 9007199254740992$ (number of n-bit numbers)

$B = 3876543219876543_{10}$ (number for a particular pattern)

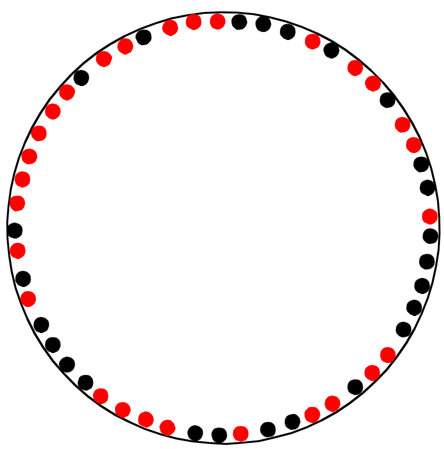
$B = 01101110001011011001000001101100100111100001010111111_2$



(dish pattern)

$B = 2388524588653369_{10}$ (another number with the same pattern)

$B = 00000110110010011110000101011111101101110001011011001_2$



(same dish pattern rotated)

How many distict plate patterns for n bits?

$n = 3$
$2^n = 8$

000
001
010
011
100
101
110
111

001 010 100
011 110 101 } S sets, $S = 2$
000
111

$n = 5$
$2^n = 32$

00000
00001
00010
00011
00100
00101
00110
00111
01000
01001
01010
01011
01100
01101
01110
01111
10000
10001
10010
10011
10100
10101
10110
10111
11000
11001
11010
11011
11100
11101
11110
11111

00001 00010 00100 01000 10000
00011 00110 01100 11000 10001
00101 01010 10100 01001 10010
00111 01110 11100 11001 10011
01011 10110 01101 11010 10101
01111 11110 11101 11011 10111 } S sets, $S = 6$
00000
11111

C distinct circular codes
$C = S + 2 = 4$

but $S \cdot n + 2 = 2^n$

$$S = \frac{2^n - 2}{n} \qquad C = \frac{2^n - 2}{n} + 2 = 4$$

C distinct circular codes
$C = S + 2 = 6$

but $S \cdot n + 2 = 2^n$

$$S = \frac{2^n - 2}{n} \qquad C = \frac{2^n - 2}{n} + 2 = 8$$

$n = 4$
$2^n = 16$

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

0001 0010 0100 1000
0011 0110 1100 1001
0101 1010
0111 1110 1101 1011
0000
1111

C distinct circular codes
$C = 6$

$$C \neq \frac{2^n - 2}{n} + 2 = 5.5$$

Why doesn't the formula work here?

Because $n = 4$ is not prime.

Patterns 0101 and 1010 are identical after a rotation of only 2, which is a factor of 4.

Plate patterns revisited

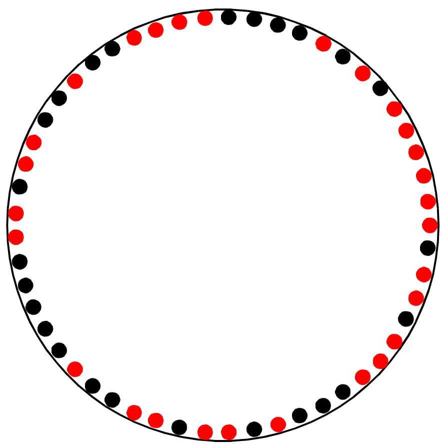
$n = 53$ (prime number of bits)

$2^n = 9007199254740992$ (number of n-bit numbers)

$$C = \frac{2^n - 2}{n} + 2 = 169947155749832$$ (number of distinct plate patterns—20,000 for every person on Earth)

$B = 3876543219876543_{10}$ (number for a particular pattern—one of n with the same pattern)

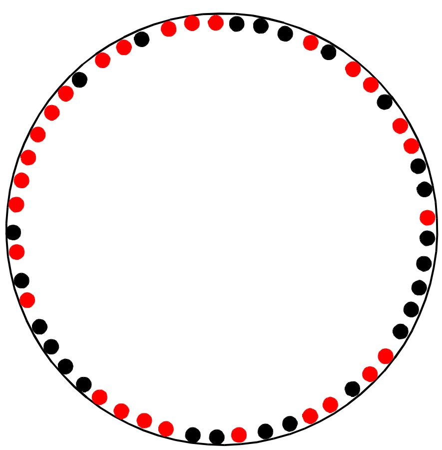
$B = 01101110001011011001000001101100100111100001010111111_2$



(dish pattern)

$B = 238852458865369_{10}$ (another number with the same pattern—smallest of the n numbers)

$B = 00000110110010011110000101011111101101110001011011001^2$



(same dish pattern rotated)

Generalization for more than 2 colors

Let the number of colors be a. For example, for $a = 4$, the colors could be magenta, cyan, yellow, and black.

Then the number of circular codes is $C = \dfrac{a^n - a}{n} + a = \dfrac{4^n - 4}{n} + 4$ for n prime.

Fermat's Little Theorem

Suppose we don't allow codes with only one color. Then for a colors, the number of circular codes (with n spots) is

$$\frac{a^n - a}{n} \quad \text{which is an integer, for } n \text{ prime.}$$

We'll call this the "Circular Code Theorem."

We can factor out an a in the numerator:

$$\frac{a^n - a}{n} = \frac{(a^{n-1} - 1)\cdot a}{n}$$

Suppose that n is not a factor of a. Then n must be a factor of $(a^{n-1} + 1)$,

and $\dfrac{a^{n-1} - 1}{n}$ is an integer for n prime.

or

$$a^{n-1} - 1 \mod n = 0 \quad \text{for } n \text{ prime}$$

which is Fermat's Little Theorem.