Playing Around

(An example of understanding an idea through examples, visualization, and analogy)

Problem as presented:

I. Traffic flow basics

A. Three characteristics are used to describe the flow of traffic: flow (or volume), velocity, and density.

1. *Flow* (q) = hourly rate of vehicles passing a point (veh/hr)
2. *Mean velocity* ($\bar{u}_s$) = harmonic mean of the velocities of vehicles
3. *Density* (k) = number of vehicles traveling over a unit length of highway at an instant in time—usually in veh/mi

B. General equation relating these characteristics: $q = k\bar{u}_s$

II. Interrupted flow

A. Things that disrupt flow: intersections, on/off ramps, changes in number of lanes, school zones, construction, crashes

B. We can describe the backup of the queue as a wave moving at a velocity u_w.

$$u_w = \frac{q_2 - q_1}{k_2 - k_1}$$

C. Example: 1600 veh/hr approach a signal-controlled intersection at 25 mph and a density of 76 veh/mi. When the signal turns red, what happens to the flow, velocity, and density of the traffic?

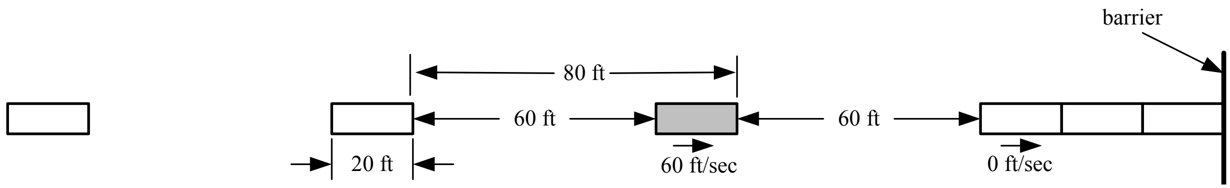
- The flow decreases from 1600 veh/hr to 0 veh/hr, a change of –1600 veh/hr.
- The velocity decreases from 25 mph to 0 mph.
- The density increases from 76 veh/mi to some maximum density. Let's assume the maximum density is 116 veh/mi—then we have a change of 40 veh/mi. The arriving vehicles form a queue, or line of stopped vehicles, at the intersection.

$$u_w = \frac{q_2 - q_1}{k_2 - k_1} = \frac{0 - 1600 \text{ veh/hr}}{116 - 76 \text{ veh/mi}} = -40 \text{ mi/hr}$$

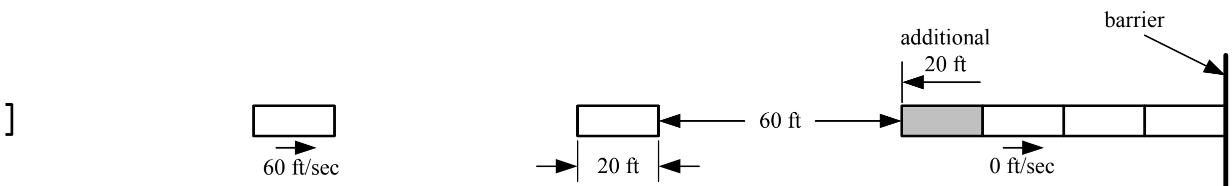
This is a stopping wave, moving backwards (shown by the negative sign) at 40 mph.

Playing around: (The following tries to record my thought process in attempting to understand the problem better. Mental notes to go back and check out something more rigorously are indicated by brackets [ ].)

Well, I'm sure the equation is right, but it's not immediately obvious. I'll try to solve an example in my own way and see if the answer agrees with the equation. In order to visualize the "maximum density," I'll assume the cars are 20 ft long and come to a stop bumper-to-bumper. If I'm going to use feet here, I'd better be consistent (to avoid conversion factors) and use ft/sec for velocity. So let's pick a problem with the cars traveling at 60 ft/sec and with 60 feet between cars (80 from front bumper to front bumper). Make a sketch; keep it simple with one lane and all moving cars the same velocity.



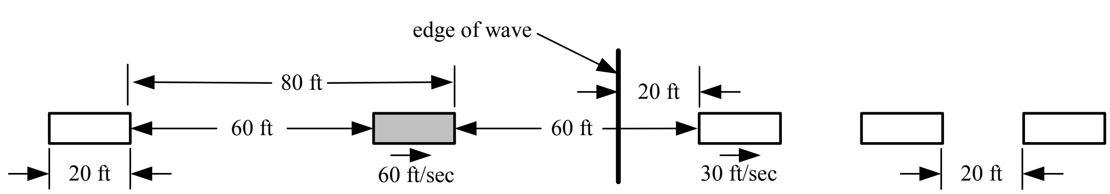
How does the wave (moving backwards) of stopped cars form? When a driver moving at 60 ft/sec sees a line a stopped cars ahead of him he starts slowing down and gradually comes to a stop (with his bumper touching the bumper of the car ahead, as assumed here). Let's say that at the moment he touches, the line is 20 feet longer. But it's too hard to calculate the time to add the 20 feet to the line if the car slows down gradually. To keep it simple, let's assume the car stays at 60 ft/sec until he touches, and then stops instantly (some deceleration!). [I don't think this assumption affects the problem; check later.] So it takes the car 1 second to close the 60-foot gap and join the line of stopped cars. The line has grown 20 feet in 1 second, so the wave (of stopped cars) is moving backwards at **20 ft/sec**.



Now I can check my result against the equation for u_w, which involves flows q and densities k. Unfortunately, my diagram doesn't have a length or area proportional to density; I can see spacing as proportional to a length, so I probably want to change the equation to involve spacing (rather than density) to get insight here. It looks like density is inversely proportional to spacing. [Pretty sure, but check later.] I can't see flow directly either. As long as the density is constant, the flow is proportional to the velocity, but the density changes.

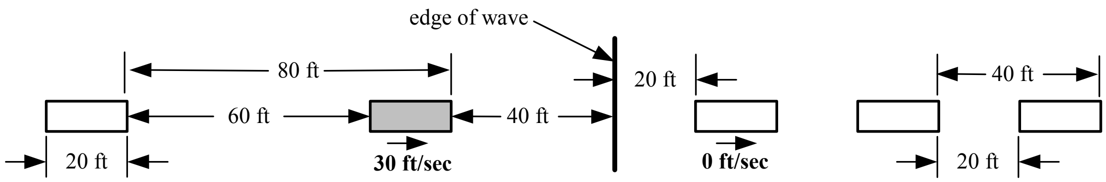
The equation involves flow in q_1 and flow out q_2, but this simple example has a flow out of $q_2 = 0$, so it's hard to see its effect. Before rearranging the equation to involve spacing, let's try an example with a nonzero flow out. [Check the agreement of 20 ft/sec with the equation later too.]

Let's see what the wave velocity is if the cars just slow down rather than stop. Let the velocity after slowing down be 30 ft/sec—only half the original velocity. What will the spacing be? Well, the cars will probably be at half the spacing (twice the density). I have to be careful and take "spacing" here to mean front bumper to front bumper. That was originally 80 feet, so we're using a final spacing of 40 feet (20 feet between cars). Draw a diagram:

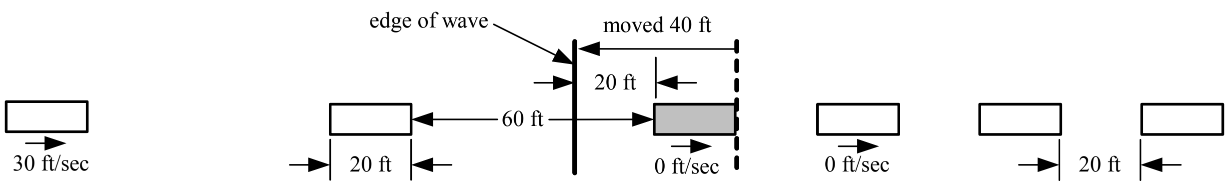


Assume that a car, seeing the line of slowed traffic, continues at 60 ft/sec until he is 20 ft from the car ahead, and then instantly slows to 30 ft/sec. I can think of it as the same situation as with the stopped cars bumper-to-bumper, but now cars have a 20-foot pole sticking out the back to hold off cars behind. Where is the edge of my wave now? If we include that pole, let's say it's 20 feet behind the last car going 30 ft/sec.

How fast does this edge of the wave move backwards due to cars slowing? I can see that now each car joining the slowed line adds 40 feet to the line, but how long does that take? And now the slowed line is moving at the same time. To simplify the visualization, let me change my frame of reference to be moving to the right at 30 ft/sec. That way the slowed line looks like it's stopped, and I'm solving the same type of problem as before. But I have to remember to subtract 30 ft/sec from the wave velocity I get because of my moving frame of reference. Draw the diagram as seen from the moving frame of reference:

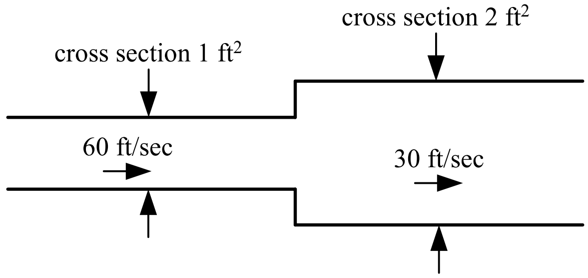


After a time of (40 ft)/(30 ft/sec) = $^{4}/_{3}$ sec, the next car (gray) will have "stopped" 20 feet from the car ahead, and the edge of the wave has moved 40 feet to the left. So the wave velocity appears to be (40 ft) /($^{4}/_{3}$ sec) = 30 ft/sec from this reference frame.



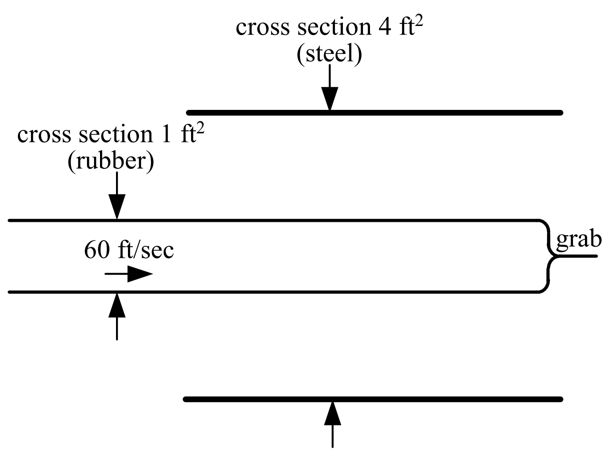
But if I subtract out the 30 ft/sec of the reference frame's movement, I'm left with an actual wave velocity of 0! What happened here? Slowing down can't always result in zero wave velocity. I had two free choices for the slowed line: the final velocity and the final spacing. There's nothing magic about picking half of the original velocity. But I realize now that I picked the final spacing so the flow would be the same. This might be desirable in traffic to keep things moving, but it's not necessary. So it seems that it's a reduced *flow* rather than a reduced velocity that causes a backward-moving wave of slowed traffic.

This slowing but keeping the same flow feels like something I've seen before. In fluid dynamics (I think it was in studying Bernoulli's principle) the velocity of a liquid in a pipe slows to half if the cross section of the pipe doubles. [Bernoulli's principle was something about pressure, but I don't think we're concerned with that in the analogy here. Go back and review the principle later.] Draw a diagram of the analogy:

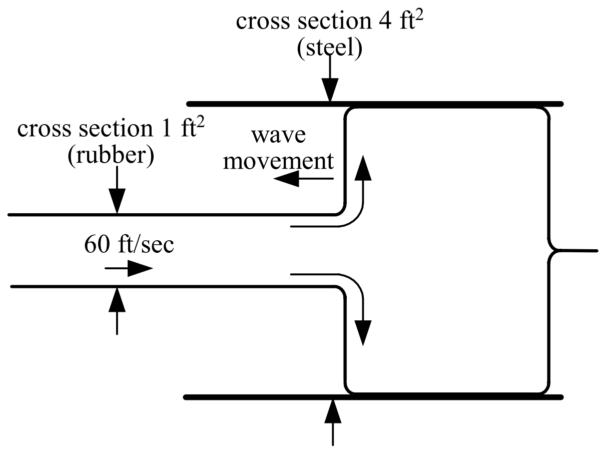


(I was thinking in three dimensions, but the picture is shone flattened here so the width of the pipe is proportional to the cross section.) It's clear from this analogy that if the velocity halves and the flow stays the same, then the "edge of the wave" (the point of the pipe's enlargement) doesn't move. What happened to the concept of car spacing? In the analogy, I have a wider flow but the same spacing of molecules, while in the original problem I had a constant single lane but compressed traffic. (I remember that an "incompressible fluid" is one of the assumptions you must state in fluid dynamics. Well, let's assume it here in the analogy so we're not dealing with too many variables.) But if I think in terms of density, nothing has changed between the original problem and the analogy. Originally, density was cars per foot, and here it's cubic feet per foot. In the original problem the density went from *one* car per 80 feet to *two* cars per 80 feet. In the analogy the density goes from 1 ft^3/ft to 2 ft^3/ft. So I don't think this change to an incompressible fluid changes anything with respect to studying wave velocity. [Move on for now, but think about this some more later.] One advantage is that now I can see something proportional to the density *k* in the equation: the cross section of the pipe. This should help with visualization and a feel for the dynamics. I can see the relation between velocity and flow a little more easily too.

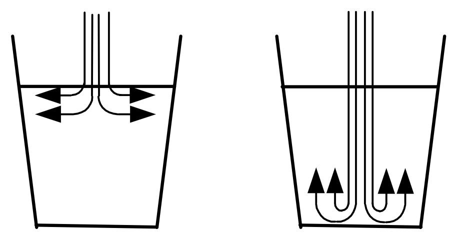
If I apply the analogy of water in a pipe to the case of the stopped traffic (which produced a nonzero wave velocity), what will I see? I can grab the pipe and suddenly cut off the flow. I'd better make the pipe flexible (rubber) to be able to squeeze it and to provide a way for the water to get backed up. In the original problem, the length of the cars determined the maximum density when they came to a stop. What can we use for a similar feel in the analogy? Since density is proportional to the cross section, let a steel pipe around the rubber pipe limit its maximum cross section. In the original problem, the density went from *one* car per 80 feet to *four* cars per 80 feet when stopped. So let the steel pipe have a cross section of 4 ft^2 (four times the 1-ft^2 cross section of the rubber pipe). Draw a diagram of the analogy (at the moment the rubber pipe is grabbed):



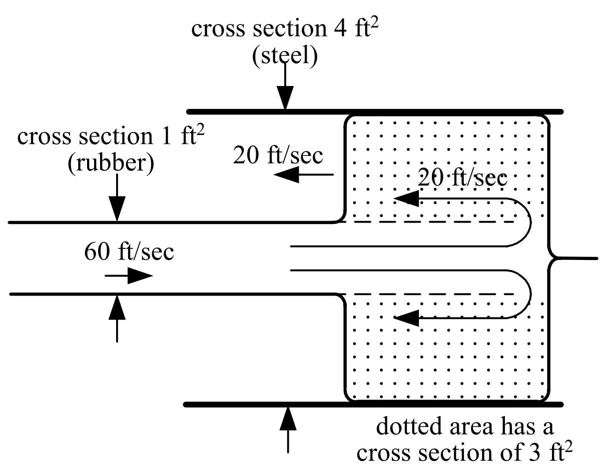
To make a wave growing from right to left, I want the water to expand the right end of the tube first. I don't have to say *why* it does this; I can just specify it. Maybe the momentum of the water causes it. So I'll picture a square bulge at the right end to receive the incoming flow:



How does the water fill the expansion? Does new water encounter stopped water and squeeze in at the very left of the bulge (like traffic would)? But I have the feeling that filling the bulge doesn't depend on how the water flows inside the bulge. When a pail of water is filled with a constant flow, the surface rises just as fast whether the new water hits the surface and goes out flat, or perhaps plunges to the bottom and swirls up the sides (see illustration).

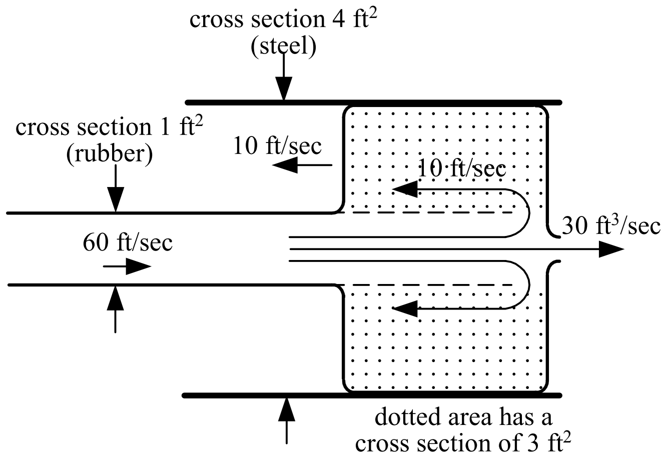


I would rather have the motion of the water filling the bulge go from right to left, because that's the direction the wave moves. It should lead to a better feel for its velocity. So imagine that the water continues at 60 ft/sec and a cross section of 1 ft^2 until it hits the end of the tube. Then it turns around, and fills the remaining 3 ft^2 of the 4-ft^2 cross section (see illustration).



It seems clear that the flow of 60 ft/sec through 1 ft^2, as it spreads out to fill the 3-ft^2 cross section coming back, will reduce to 20 ft/sec (one third of the 60 ft/sec). Perhaps I should be checking some equation here relating velocity, cross section, and flow, but my physical sense says it's right. So the wave velocity is **20 ft/sec**, which is the same answer I got for the stopped traffic, but I have better insight with this model.

I still haven't checked against the equation, but before I do, I want to work with a nonzero output flow in this water analogy. I want to release my grip on the tube at the right end a little and let out some flow. Again, I have two parameters to decide on: the output velocity and cross section (density). But I feel that it's the flow that matters, robbing some of the input flow (60 ft/sec × 1 ft^2 = 60 ft^3/sec) from back-filling the 3-ft^2 part of the bulge's cross section. Let's say I let out a flow of half this—or 30 ft^3/sec—at the right. It doesn't matter what cross section I choose, so I'll just show the flow rate of 30 ft^3/sec on the diagram.



Since this leaves only the other half of the input flow to fill the bulge, it will fill it at half the previous rate and velocity, for a wave velocity of **10 ft/sec** rather than 20 ft/sec. So now I can feel why the equation deals with the difference $q_1 - q_2$ of the input flow and output flow, and why it deals with the difference $k_2 - k_1$ of the final and initial densities. The difference in density is like* the difference in cross section that gets filled by the difference in flows. In the case of the water analogy, $q_1 - q_2 = 60 - 30 = 30$ ft^3/sec, and $k_2 - k_1 = 4 - 1 = 3$ ft^3/ft. So the wave velocity is

$$u_w = \frac{q_1 - q_2}{k_2 - k_1} = \frac{30 \text{ ft}^3/\text{sec}}{3 \text{ ft}^3/\text{ft}} = 10 \text{ ft/sec}$$

This equation is different from the one given at the beginning by a minus sign, but I see that I have defined a positive wave velocity as being to the left. Scientist like things neat and tidy with all velocities (both flow velocity and wave velocity) defined as positive in the same direction. But negative numbers usually get in the way of intuition and feeling.

I have to believe that the person generating the equation at the beginning had some insight before he wrote it down. But it's obviously much quicker to just give the equation than to provide the insight (it took six pages here). And yet, is the equation the essence of engineering?

* Going back later, I find that cross section in ft^2 is not just proportional to density, it *is* density in ft^3/ft.